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COMMON FIXED POINTS OF TWO PAIRS OF GENERALIZED WEAKLY CONTRACTIVE MAPS

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Abstract

The theorem of this paper is a generalization of a result of Babu et al. [1], as well as some other theorems in the literature.

1. Introduction

In a recent paper [1], the authors proved a fixed point theorem for four maps satisfying a generalized weakly contractive condition. In this paper, a substantial generalization of their result is established.

Let A, B, S, T be selfmaps of a metric space X. These maps will be called *generalized weakly contractive*, if

$$\psi(d(Ax, By)) \leq \psi(m(x, y)) - \varphi(d(Ax, By)), \quad \text{for all } x, y \in X, \tag{1}$$

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where

$$m(x, y) := \max\{d(Ax, By), d(Ax, Sx), d(By, Ty),$$
$$[d(Ax, By) + d(By, Sx)]/2\}, \tag{2}$$

and where $\psi : [0, \infty) \to [0, \infty)$, ψ is continuous, monotone increasing, and satisfies $\psi(t) < t$ for each $t > 0$, and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$.

A pair of maps A, S is said to be *weakly compatible*, if they commute at coincidence points.

Theorem 1. *Let A, B, S, T be selfmaps of a complete metric space (X, d), which satisfy $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and* (1). *If the pairs (A, S) and (B, T) are weakly compatible and one of the ranges $A(X), B(X), S(X), T(X)$ is closed, then A, B, S, T have a unique common fixed point.*

Proof. For any $x_0 \in X$, there exist an $x_1 \in X$ such that $y_0 := Ax_0 = Tx_1$. Similarly, there exists a point $x_2 \in X$ such that $y_1 := Bx_1 = Sx_2$. In general, $\{y_n\}$ is defined by $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$.

From (1),

$$\psi(d(Ax_{2n}, Bx_{2n+1})) \le \psi(m(x_{2n}, x_{2n+1})) - \varphi(d(Ax_{2n}, Bx_{2n+1})).$$

From (2),

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad [d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})]/2\} \\ &= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \\ &\quad [0 + d(y_{2n+1}, y_{2n-1})]/2\} \\ &= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}, \end{aligned}$$

and

$$\psi(d(y_{2n},\, y_{2n+1})) \leq \psi(\max\{d(y_{2n},\, y_{2n+1}),\, d(y_{2n},\, y_{2n-1})\})$$

$$- \varphi(d(y_{2n},\, y_{2n+1})).$$

In a similar manner, it can be shown that

$$\psi(d(y_{2n+1},\, y_{2n+2})) \leq \psi(\max\{d(y_{2n+1},\, y_{2n+2}),\, d(y_{2n+1},\, y_{2n})\})$$

$$- \varphi(d(y_{2n+1},\, y_{2n+2})).$$

Thus, for all n,

$$\psi(d(y_n,\, y_{n+1})) \leq \psi(\max\{d(y_n,\, y_{n+1}),\, d(y_n,\, y_{n-1})\}) - \varphi(d(y_n,\, y_{n+1})). \quad (3)$$

If there exists an n for which $d(y_n,\, y_{n-1}) < d(y_n,\, y_{n+1})$, it follows from (3) that

$$\psi(d(y_n,\, y_{n+1})) \leq \psi(d(y_n,\, y_{n+1})) - \varphi(d(y_n,\, y_{n+1})) < \psi(d(y_n,\, y_{n+1})),$$

a contradiction.

Therefore, for all n, we have

$$d(y_n,\, y_{n+1}) \leq d(y_{n-1},\, y_n),$$

and $\{d(y_n,\, y_{n+1})\}$ is a positive nonincreasing sequence and therefore converges to a limit $a \geq 0$.

Suppose that $a > 0$. Then, taking the limit of (3) as $n \to \infty$, one obtains

$$\psi(a) \leq \psi(a) - \varphi(a),$$

a contradiction. Therefore $a = 0$.

We wish to show that $\{y_n\}$ is a Cauchy sequence. It will be sufficient to show that $\{y_{2n}\}$ is Cauchy. Suppose that it is not Cauchy. Then, there exists an $\epsilon > 0$ and two subsequences of even integers $\{m(k)\}$ and $\{n(k)\}$, such that $n(k) > m(k) > 2k$ and

$$d(y_{m(k)},\ y_{n(k)}) \geq \epsilon \quad \text{and} \quad d(y_{m(k)},\ y_{n(k)-2}) < \epsilon.$$

Then

$$\begin{aligned}\epsilon &\leq d(y_{m(k)},\ y_{n(k)})\\ &\leq d(y_{m(k)-1},\ y_{n(k)+1}) + d(y_{m(k)-1},\ y_{m(k)}) + d(y_{n(k)+1},\ y_{n(k)})\\ &\leq d(y_{m(k)},\ y_{n(k)}) + 2d(y_{m(k)-1},\ y_{m(k)}) + 2d(y_{n(k)+1},\ y_{n(k)}).\end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$\lim_k d(y_{m(k)-1},\ y_{n(k)+1}) = \epsilon.$$

Also,

$$\begin{aligned}d(y_{m(k)},\ y_{n(k)}) &\leq d(y_{m(k)},\ y_{n(k)+1}) + d(y_{n(k)},\ y_{n(k)+1})\\ &\leq d(y_{m(k)},\ y_{n(k)}) + 2d(y_{n(k)},\ y_{n(k)+1}).\end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$\lim_k d(y_{m(k)},\ y_{n(k)+1}) = \epsilon.$$

Using (1),

$$\begin{aligned}\psi(d(y_{m(k)},\ y_{n(k)+1})) &= \psi(d(Ax_{m(k)},\ Bx_{n(k)+1}))\\ &\leq \psi(m(x_{m(k)},\ x_{n(k)+1})) - \varphi(d(Ax_{m(k)},\ Bx_{n(k)+1})). \quad (4)\end{aligned}$$

From (2),

$$\begin{aligned}&m(x_{m(k)},\ x_{n(k)+1})\\ &= \max\{d(Ax_{m(k)},\ Bx_{n(k)+1}),\ d(Ax_{m(k)},\ Sx_{m(k)}),\ d(Bx_{n(k)+1},\ Tx_{n(k)+1}),\\ &\quad [d(Ax_{m(k)},\ Tx_{n(k)+1}) + d(Bx_{n(k)+1},\ Sx_{m(k)})]/2\}\\ &= \max\{d(y_{m(k)},\ y_{n(k)+1}),\ d(y_{m(k)},\ y_{m(k)-1}),\ d(y_{n(k)+1},\ y_{n(k)}),\\ &\quad [d(y_{m(k)},\ y_{n(k)}) + d(y_{n(k)+1},\ y_{m(k)-1})]/2\}.\end{aligned}$$

Substituting into (4) and taking the limit as $k \to \infty$, one obtains

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

a contradiction. Therefore $\{y_n\}$ is Cauchy, hence convergent, to some point $z \in X$. Thus we have

$$\lim_n Ax_{2n} = \lim_n Tx_{2n+1} = z,$$

and

$$\lim_n Bx_{2n+1} = \lim_n Tx_{2n+2} = z.$$

Suppose that $S(X)$ is closed. Then, there exists a $u \in X$ such that $z = Su$.

We claim that $Au = z$. Suppose not. Then, from (1),

$$\psi(d(Au, Bx_{2n+1})) \leq \psi(m(u, x_{2n+1})) - \varphi(d(Au, Bx_{2n+1})).$$

From (2),

$$m(u, x_{2n+1}) = \max\{d(Su, Tx_{2n+1}), d(Su, Au), d(Tx_{2n+1}, Bx_{2n+1}),$$
$$[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)]/2\}.$$

Substituting and taking the limit as $n \to \infty$ gives

$$\psi(d(Au, z)) \leq \psi(d(z, Au)) - \varphi(d(Au, z)),$$

a contradiction. Therefore, $z = Au = Su$ and u is a coincidence point of A and S.

Since A and S are weakly compatible, $Az = ASu = SAu = Sz$. We shall now show that z is a common fixed point of A and S. If $Az \neq z$, then, from (1),

$$\psi(d(Az, Bx_{2n+1})) \leq \psi(m(z, x_{2n+1})) - \varphi(d(Az, Bx_{2n+1})).$$

Taking the limit as $n \to \infty$ yields

$$\psi(d(Az, z)) \leq \psi(d(Az, z)) - \varphi(d(Az, z)),$$

a contradiction. Therefore $z = Az = Sz$.

Since $A(X) \subseteq T(X)$, there is a point $v \in X$ such that $Az = Tv$. Thus $Az = Tv = Sz = z$.

Suppose that $Bv \neq z$. Then, from (1),

$$\psi(d(z, Bv)) = \psi(d(Az, Bv)) \leq \psi(d(m(z, v))) - \varphi(d(Az, Bv)),$$

which leads to

$$\psi(d(z, Bv)) \leq \psi(d(z, Bv)) - \varphi(d(z, Bv)),$$

which is a contradiction. Therefore $Bv = Tv = z$. Since the maps are weakly compatible, $Bz = BTv = TBv = Tz$, and z is a coincidence point for T and B.

Suppose that $Bz \neq z$. Then, from (1),

$$\psi(d(z, Bz)) = \psi(d(Az, Bz)) \leq \psi(m(z, z)) - \varphi(d(Az, Bz)),$$

which leads to

$$\psi(d(z, Bz)) \leq \psi(d(z, Bz)) - \varphi(d(z, Bz)),$$

a contradiction. Therefore $Az = Bz = Sz = Tz = z$, and z is a common fixed point.

To show uniqueness, suppose that w is also a common fixed point, with $z \neq w$. Using (1),

$$\psi(d(z, w)) = \psi(d(Az, Bw)) \leq \psi(m(z, ws)) - \varphi(d(Az, Bw)),$$

which leads to

$$\psi(d(z, w)) \leq \psi(d(z, w)) - \varphi(d(z, w)),$$

a contradiction. Therefore, the common fixed point is unique.

The proofs, assuming that $A(X)$, $S(X)$, or $T(X)$ is closed are similar.

$\square$

Special cases of Theorem 1 appear in the references listed in the bibliography.

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